IMPLICATIVITY AND IRREDUCIBILITY IN ORTHOMODULAR LATTICES

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CHAPTER I

PRELIMINARIES

1. Introduction

The current model for the "logic" of (non-relativistic) quantum mechanics is the lattice of closed subspaces of a separable, infinite dimensional, Hilbert space. However, as Mackey (5) points out, this assumption is rather *ad hoc* in character. One would like to be able to deduce such an assumption from a list of physically plausible assumptions. Now one can exhibit some quite convincing arguments to show that the "logic" of any experimental theory ought to be an orthocomplete, orthomodular, poset: perhaps even an orthomodular lattice (7). From this point, however, it is not clear how to transfer various physical assumptions into suitable lattice theoretic statements and thereby deduce the "correct logic" for quantum mechanics. What appears to be needed is a list of physically interpretable lattice theoretic definitions. One possible candidate for such a list is the topic of the present work, namely, the notion of an *implicative pair*. This and some of its consequences will be treated in Chapter 2.

Let us make it clear at the outset that we do not intend to engage in polemics regarding the physical interpretations (if any) of our work. Such discussions must await subsequent research on the connection between lattice theory and physics. Hence, our approach will be from a formal mathematical point of view.

2. Basic definitions and theorems.

In this section we give some of the basic, well-known, theorems and definitions found in the field of *orthomodular lattice* theory. We will not present proofs since they are readily available in (2).

Definition 1.2.1. An *orthomodular lattice* is a lattice L having a 0 and a 1, and equipped with an orthocomplementation $':L \longrightarrow L$ such that the *orthomodular identity* is satisfied:

$$x \leq y \text{ implies } y = x \vee (y \wedge x').$$

Convention 1.2.2. Throughout the rest of this work, L will always represent an orthomodular lattice.

Definition 1.2.3. Let $e,f \in L$ with $e \leq f$. We then define the *interval* $L(e,f)$ by: $L(e,f) = \{x \in L : e \leq x \leq f\}$.

Lemma 1.2.4. If $e,f \in L$ with $e \leq f$, then $L(e,f)$ is itself an orthomodular lattice with orthocomplementation

$$x \longrightarrow x\# = e \vee (f \wedge x') = (e \vee x') \wedge f.$$

Definition 1.2.5. Let $e \in L$. We then define a mapping $\emptyset_e : L \longrightarrow L$, called the *Sasaki projection determined by e*, by the rule: $x\emptyset_e = e \wedge (x \vee e')$.

Lemma 1.2.6. Let $e \in L$. Then the following hold.

(i) $\emptyset_e = \emptyset_e^2$.

(ii) $(x \vee y)\emptyset_e = x\emptyset_e \vee y\emptyset_e$.

(iii) $x\emptyset_e = x$ if and only if $x \leq e$.

(iv) $x\emptyset_e = 0$ if and only if $x \leq e'$.

Definition 1.2.7. Let $e,f \in L$. Then f is said to commute with e, in symbols fCe, if and only if

$$f\emptyset_e = f \wedge e.$$

We will write $f \not{C} e$ providing f does not commute with e.

Lemma 1.2.8. Let $e,f,g \in L$. Then the following hold.

(i) $f \leq e$ implies fCe.

(ii) fCe implies eCf.

(iii) fCe implies fCe'.

(iv) If any two of the three conditions eCf, fCg, or eCg hold, then $(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$ and $(e \wedge f) \vee g = (e \vee g) \wedge (f \vee g)$.

(v) If fCe_α for each $\alpha \in A$, and if $\bigvee_{\alpha \in A} e_\alpha$ exists, then $fC \bigvee_{\alpha \in A} e_\alpha$.

Definition 1.2.9. Let $A \subset L$. Then we define

$$C(A) = \{x \in L : xCy \text{ for every } y \in A\}.$$

If $A = L$ then the set $C(L)$ is called the center of L. If A is a singleton set, say $\{a\}$, we will use the abusive notation $C(a)$ for $C(\{a\})$. L is said to be irreducible if and only if $C(L) = \{0,1\}$. If $e \in C(L)$, $e \neq 0$, and $e \neq 1$,

then one can prove that L can be "factored" into the Cartesian product: $L = L(0,e) \times L(0,e')$, hence the reason for the word "irreducible".

Lemma 1.2.10. Let $a \in L$. Then the following hold.

(i) $C(a) = \{(x \vee a) \wedge (x \vee a') : x \in L\}$.

(ii) $C(a) = \{(x \wedge a) \vee (x \wedge a') : x \in L\}$.

(iii) $C(a) = \{x\emptyset_a \vee x\emptyset_{a'} : x \in L\}$.

3. The atomic bisection property.

Throughout this section we will follow the treatment given by Janowitz (3).

Definition 1.3.1. L is said to be atomic if and only if $x \in L$ and $x \neq 0$ imply there exists an atom e such that $e \leq x$.

Definition 1.3.2. L is said to have the atomic bisection property if and only if L is atomic and for every pair of atoms $b,c \in L$, there exists an atom $a \in L$ such that $a \nleq b$, $a \nleq c$, and $a \leq b \vee c$.

Remark. 1.3.3. In definition 1.3.2, it suffices to suppose that b and c are orthogonal atoms, i.e. $b \leq c'$. One can easily see this by noting the identity

$$b \vee c = b \vee [(b \vee c) \wedge b'].$$

Lemma 1.3.4. If L has the atomic bisection property, then L is irreducible.

The proof of this lemma is omitted since it can be obtained as a consequence of theorems 2.3.2 and 2.3.3 of Chapter II.

Lemma 1.3.5. If L has the atomic bisection property, then so does any interval L(e,f).

Proof. Note that the mapping $a \longrightarrow a \wedge e'$ is an orthoisomorphism of L(e,f) onto $L(0, f \wedge e')$. Thus, since $L(0, f \wedge e')$ has the atomic bisection property, so does L(e,f).

Lemma 1.3.6. If L is atomic, complete, and modular, then the irreducibility of L is both a necessary and sufficient condition for L to have the atomic bisection property.

Proof. According to Kaplansky (4), every complete, modular, orthocomplemented lattice is a continuous geometry. The result now follows from (6, p. 80, Th. 2.4).

Theorem 1.3.7. Let L be atomic. Then the following conditions are mutually equivalent.

(i) L has the atomic bisection property.

(ii) Every interval L(e,f) is irreducible.

(iii) Every interval L(0,a) is irreducible.

If in addition, L is complete and modular, we can add

(iv) L is irreducible.

4. The relations ∇ and S.

In this section we introduce the relations ∇ and S on an orthomodular lattice. The relation ∇ has been studied by Maeda (6) for arbitrary lattices and by S. Holland, Jr. and M. F. Janowitz for orthomodular lattices. The relation S was apparently first defined by S. Holland, Jr.

Definition 1.4.1. Let $e,f \in L$.

(i) $eNCf$ if and only if $e \wedge f = e \wedge f' = 0$.

(ii) We say that e and f are detached, in symbols $e \nabla f$, if and only if e and f are orthogonal and whenever $g \in L$ and $gNCe$, then gCf.

(iii) We say that e and f are separated, in symbols eSf, if and only if e and f are detached in the interval $L(0, e \vee f)$.

Lemma 1.4.2. Let $e,f \in L$. Then $eNCf$ if and only if every non-zero subelement of e fails to commute with f.

Proof. Let $e,f \in L$ and suppose that $eNCf$. Then by definition, $e \wedge f = e \wedge f' = 0$. Let $0 \neq e_1 \leq e$. Then, clearly, $e_1 \wedge f = e_1 \wedge f' = 0$. If e_1Cf, then $e_1 = (e_1 \wedge f) \vee (e_1 \wedge f') = 0$. But $e_1 \neq 0$, so this is a contradiction. Hence, e_1 does not commute with f.

Conversely, suppose every non-zero subelement of e fails to commute with f. $e \wedge f$ and $e \wedge f'$ are both subelements of e and both commute with f. Hence, $e \wedge f = e \wedge f' = 0$. By definition, then, $eNCf$.

Theorem 1.4.3. (Holland). Let $e,f \in L$. Then there are uniquely determined elements e_1 and e_2 in L such that e_1 is orthogonal to e_2, e_1Cf, e_2NCf, and $e = e_1 \vee e_2$. In fact $e_1 = (e \wedge f) \vee (e \wedge f')$ and $e_2 = (f'\emptyset_e) \wedge (f\emptyset_e)$.

Proof. Note that $(e \wedge f) \vee (e \wedge f') \le ((f \wedge e) \vee e') \vee ((f' \wedge e) \vee e')$ so that e_1 is orthogonal to e_2. Clearly $e_1 C f$. To show that $e_2 NC f$ we compute:

$e_2 \wedge f = [(f' \vee e') \wedge e] \wedge [(f \vee e') \wedge e] \wedge f =$

$[(f' \vee e') \wedge e] \wedge [f \wedge e] = [f' \vee e'] \wedge [f \wedge e] = 0$;

$e_2 \wedge f' = [(f' \vee e') \wedge e] \wedge f' \wedge [(f \vee e') \wedge e] =$

$[f' \wedge e] \wedge [(f \vee e') \wedge e] = [f' \wedge e] \wedge [f \vee e'] = 0.$

Hence $e_2 NC f$. Since $e_2 = e \wedge e_1'$ and $e_1 \le e$, we have $e = e_1 \vee (e \wedge e_1') = (e_1 \vee e_2)$. It remains to show uniqueness. Suppose g_1 and g_2 have the same respective properties as e_1 and e_2. Then $e_1 = (e \wedge f) \vee (e \wedge f') = [(g_1 \vee g_2) \wedge f] \vee [(g_1 \vee g_2) \wedge f'] = (g_1 \wedge f) \vee (g_1 \wedge f') = g_1$, i.e., $g_1 = e_1$. Hence, $e_2 = e \wedge e_1' = e \wedge g_1' = g_1 \vee g_2) \wedge g_1' = g_2 \wedge g_1' = g_2$.

Definition 1.4.4. Let $e,f \in L$ and let e_1, e_2 be as in theorem 4.3. We shall refer to $e = e_1 \vee e_2$ as the C-NC decomposition of e with respect to f.

Theorem 1.4.5. (Holland-Janowitz) Let $e,f \in L$. Then the following are equivalent.

(i) $e \nabla f$.

(ii) For all $g \in L$, $g \wedge e' = 0$ implies that f is orthogonal to g.

(iii) For all $g \in L$, $g \vee e = 1$ implies $f \le g$.

(iv) If g is any complement of e in L, then $f \le g$.

(v) For all $g \in L$, f is orthogonal to $e\emptyset_g$.

(vi) For all $g \in L$, $g = (e \vee g) \wedge (f \vee g)$.

(vii) For all $g \in L$, $f \leq e \vee g$ implies $f \leq g$.

<u>Proof</u>. <u>(i) implies (ii)</u>. Suppose that $e \nabla f$ and that $g \in L$ with $g \wedge e' = 0$. Let $g = g_1 \vee g_2$ be the C-NC decomposition of g with respect to e. Then $g_1 = (g \wedge e) \vee (g \wedge e') = g \wedge e$ and $g_2 = g \wedge g_1 = g \wedge (g' \vee e')$. Since $e \nabla f$ and $g_2 NCe$, we have by definition that $g_2 Cf$. Also, since e is orthogonal to f we have g_1 orthogonal to f. Hence, $f \wedge g = f \wedge (g_1 \vee g_2) = (f \wedge g_1) \vee (f \wedge g_2) = f \wedge g_2 = f \wedge g \wedge (g' \vee e') = (f \wedge g \wedge g') \vee (f \wedge g \wedge e') = 0$. Since fCg_1 and fCg_2, then fCg. But fCg and $f \wedge g = 0$ implies that f and g are orthogonal.

<u>(ii) implies (iii)</u> and <u>(iii) implies (iv)</u> are trivial.

<u>(iv) implies (v)</u>. Assume (iv) and let $h = (e' \vee g) \wedge [g' \vee (e' \wedge g)]$. Then $h \wedge e = (e' \vee g) \wedge [g' \vee (e' \wedge g)] \wedge e = (e' \vee g) \wedge [(g' \wedge e) \vee (e' \wedge g \wedge e)] = (e' \vee g) \wedge (g' \wedge e) = 0$ and $h \vee e = [e \vee (e' \vee g)]$ $[g' \vee (e' \wedge g) \vee e] = 1$. Hence h is a complement of e. Therefore $f \leq h \leq g' \vee (e' \wedge g) = (e\emptyset_g)'$.

<u>(v) implies (vi)</u>. Put $h = (e \vee g) \wedge (f \vee g)$. Since $g < h$, it suffices to prove $h \wedge g' = 0$. But this follows easily: $h \wedge g' = (e\emptyset_{g'}) \wedge (f \vee g) \leq f' \wedge g' \wedge (f \vee g) = 0$.

<u>(vi) implies (vii)</u>. $g = (e \vee g) \wedge (f \vee g)$ implies $g \wedge f = (e \vee g) \wedge (f \vee g) \wedge f$ which in turn implies $g \wedge f = (e \vee g) \wedge f$.

<u>(vii) implies viii)</u>. $f \leq e \vee g$ implies $(e \vee g) \wedge f = f$. By (vii), $g \wedge f = (e \vee g) \wedge f = f$, so that $f \leq g$.

<u>(viii) implies (i)</u>. First put $g = e'$ in (viii) and obtain f orthogonal to e. Now suppose hNCe. We must prove that hCf. Since hNCe, then $h \wedge e' = 0$ so that $e \vee h' = 1$. By (viii), $f \leq h'$, hence fCh.

<u>Corollary</u> 1.4.6.

(i) The relation ∇ on any orthomodular lattice is symmetric.

(ii) $e \in C(L)$ if and only if $e \nabla e'$.

(iii) Let $\{e_\alpha\}$ be a family of elements of L with $e = V_\alpha\{e_\alpha\}$. If $f \in L$ and if $e_\alpha \nabla f$ for all α, then $e \nabla f$.

<u>Theorem</u> 1.4.7. Let $e,f \in L$. Then the following statements are equivalent.

(i) eSf.

(ii) For all $g \leq e \vee f$, $g \wedge e' = 0$ implies f orthogonal to g.

(iii) For all $g \leq e \vee f$, $g \vee e = f \vee e$ implies $f \leq g$.

(iv) If $g \leq e \vee f$ and g is a complement of e in $L(0,e \vee f)$, then $f \leq g$.

(v) If $g \leq e \vee f$, then f and $e\emptyset_g$ are orthogonal.

(vi) If $g \leq e \vee f$, then $g = (e \vee g) \wedge (f \vee g)$.

(vii) If $g \leq e \vee f$, then $(e \vee g) \wedge f = g \wedge f$.

(viii) If $f \leq e \vee g \leq e \vee f$, then $f \leq g$.

Proof. Follows at once from 4.6 and definition 4.1.

Corollary 1.4.8.

(i) The relation S on any orthomodular lattice is symmetric.

(ii) Let $e,f \in L$ with $e < f$. Then e is central in $L(0,f)$ if and only if $eSe' \wedge f$.

Theorem 1.4.9. Let $e,f \in L$ with $e \wedge f = 0$. Then the following statements are equivalent.

(i) esf.

(ii) For all $g \in L$, $g \leq e \vee f$, $g \wedge e = 0$, $g \wedge f = 0$ imply $g = 0$.

(iii) For all $g \in L$, $g \leq e \vee f$ implies $g = (e \wedge g) \vee (f \wedge g)$.

(iv) e central in $L(0,e \vee f)$.

(v) There is no $g \nleq f$ such that $e \wedge g = 0$ and $e \vee g = e \vee f$.

(vi) For all $g \in L$, $g \wedge (e \vee f) = (g \wedge e) \vee (g \wedge f)$.

(vii) For all $g \in L$, $gC(e \vee f)$ implies gCe and gCf.

Proof. <u>(i) implies (ii)</u>. From theorem 1.4.7, part (v), setting $g = f$, we obtain f orthogonal to $e\emptyset_f$. Therefore, $e\emptyset_f = 0 = e \wedge f$, i.e., eCf. Now let $g \in L(0,e \vee f)$ satisfy $g \wedge e = g \wedge f = 0$. By (vi) of theorem 1.4.7, $g = (e \vee g) \wedge (f \vee g)$. Now $0 = (f \wedge g) \vee (e \wedge g) = [f \wedge (e \vee g) \wedge (f \vee g)] \vee [e \wedge (e \vee g) \wedge (f \vee g)] = [(e \vee g) \wedge f] \vee [(f \vee g) \wedge e]$. Now $(e \vee g) \wedge f \leq f \vee g$ and $[(e\vee g) \wedge f]Ce$, so that $0 = [f \vee g] \wedge [((e \vee g) \wedge f) \vee e] = (f \vee g) \wedge (e \vee g) \wedge (f \vee e) = g \wedge (f \vee e) = g$

<u>(ii) implies (iii)</u>. Let $g \leq e \vee f$. Since $(e \wedge g) \vee (f \wedge g) \leq g$, it will suffice to show that $h = g \wedge (e \wedge g)' \wedge (f \wedge g)' = 0$. But $h \leq g \leq e \vee f$ and $h \wedge e = h \quad f = 0$. Hence, $h = 0$ by (ii).

<u>(iii) implies (iv)</u>. The orthocomplement of e in $L(0,e \vee f)$ is given by $e^{\#} = (e \vee f) \wedge e'$. Condition (iii) with $g = e^{\#}$ yields $e^{\#} \leq f$. Thus condition (iii) gives $g = (e \wedge g) \vee (e^{\#} \wedge g)$ for every $g \in L(0,e \vee f)$. Hence, by lemma 1.2.10, e is central in $L(0,e \vee f)$.

<u>(iv) equivalent to (v)</u> is clear.

<u>(iv) implies (vi)</u>. Since e is central in $L(0,e \vee f)$ and since $e \wedge f = 0$, then $f = e^{\#} = e' \wedge (e \vee f)$.

Given $g \in L$, set $h = g \wedge (e \vee f)$. Since h commutes with e in $L(0, e \vee f)$, then $g = (h \wedge e) \vee (h \wedge e^{\#}) \wedge (h \wedge e) \vee (h \wedge f)$. In other words, $g \wedge (e \vee f) = [g \wedge (e \vee f) \wedge e] \vee [g \wedge (e \vee f) \wedge f] = (g \wedge e) \vee (g \wedge f)$.

<u>(vi) implies (vii)</u>. By (vi) we have $g \wedge (e \vee f) = (g \wedge e) \vee (g \wedge f)$ and $g' \wedge (e \vee f) = (g' \wedge e) \vee (g' \wedge f)$. Since $gC(e \vee f)$, we have $e \vee f = [(e \vee f) \wedge g] \vee [(e \vee f) \wedge g'] = (e \wedge g) \vee (f \wedge g) \vee (e \wedge g') \vee (f \wedge g') = [(e \wedge g) \vee (e \wedge g')] \vee [(f \wedge g) \vee (f \wedge g')]$. Now in (vi), let $g = e'$, and obtain eCf. Thus, since $e \wedge f = 0$, we have that e and f are orthogonal. Hence $(e \wedge g) \vee (e \wedge g')$ is orthogonal to $(f \wedge g) \vee (f \wedge g')$. Therefore, $e = (e \vee f) \wedge e = \{[(e \wedge g) \vee (e \wedge g')] \vee [(f \wedge g) \vee (f \wedge g')]\} \wedge e = (e \wedge g) \vee (e \wedge g')$, i.e., eCg. Similarly, fCg.

<u>(vii) implies (i)</u>. Condition (vi) of theorem 1.4.7 follows from (vii) at once.

<u>Corollary</u> 1.4.10. If L is a modular orthomodular lattice, then in L we have $\nabla = S$.

CHAPTER II

MAIN RESULTS

1. Implicative Pairs

In this section we shall study the notion of an <u>implicative pair</u> in an orthomodular lattice. This notion is motivated by the study of classical logic as follows. Let B be a Boolean lattice, $e, f \in B$. We then define an element of B called "e implies f", written $e \supset f$, by the equation $e \supset f = e' \vee f$. Two of the theorems which then hold are

(1) $e \wedge (e \supset f) \leq f$, (modus ponens), and

(2) $e \wedge h \leq f$ if and only if $h \leq (e \supset f)$, (exportation).

Now in B these theorems hold for any pair of elements e and f. A reasonable question is: Can we define an operation $\supset$ in an arbitrary orthomodular lattice such that (1) and (2) hold? The answer is no, in general, since Skolem (1) has shown that any lattice L in which (1) and (2) hold for every e and f in L is necessarily distributive. This suggests that the following definition will be non-trivial for orthomodular lattices.

<u>Definition</u> 2.1.1. The elements e and f in the orthomodular lattice L are said to form an <u>implicative</u>

pair, written $I(e,f)$, if and only if there exists $g \in L$ such that

(1) $e \wedge g \leq f$ and,

(2) if $h \in L$, then $e \wedge h \leq f$ implies $h \leq g$,

both hold.

Remark 2.1.2. If $h \leq g$ and if (1) holds, then $e \wedge h \leq e \wedge g \leq f$, i.e., (1) implies the converse of (2).

Lemma 2.1.3. g in definition 2.1.1 is unique and in fact $g = e' \vee f$.

Proof. Let g_1 and g_2 both satisfy definition 2.1.1. Since $e \wedge g_1 \leq f$, we have by (2) that $g_1 \leq g_2$. Similarly, $e \wedge g_2 \leq f$ implies that $g_2 \leq g_1$. Hence $g_1 = g_2$ and so g is unique. Since $e \wedge f \leq f$, we have by (2) that $f \leq g$. Also, $e \wedge e' \leq f$ so that $e' \leq g$. Hence, $e' \vee f \leq g$. Since $e' \vee f \leq g$, it suffices to show that $e \wedge f' \wedge g = 0$. But $e \wedge g \leq f$ by (1), and so $e \wedge g \wedge f' \leq f \wedge f' = 0$. Thus $g = e' \vee f$.

Convention 2.1.4. Henceforth we shall write $e \supset f = e' \vee f$ if and only if $I(e,f)$.

Lemma 2.1.5. $I(e,f)$ implies eCf.

Proof. By lemma 2.1.3 we know that $e \wedge (e' \vee f) \leq f$, i.e., $f\emptyset_e \leq f$. Hence, $f\emptyset_e \leq f \wedge e$. But $f\emptyset_e \leq f \wedge e$ always holds so that $f\emptyset_e = f \wedge e$. By definition, eCf.

Theorem 2.1.6. The following hold in L.

(i) $I(a,c)$ and $I(b,c)$ imply $I(a \vee b,c)$. Moreover, $(a \supset c) \wedge (b \supset c) = (a \vee b) \supset c$.

(ii) $I(a,b)$ and $I(a,c)$ imply $I(a, b \wedge c)$. Moreover, $(a \supset b) \wedge (a \supset c) = a \supset (b \wedge c)$.

(iii) $I(a,b)$ and $I(b,a)$ imply $I(a \vee b, a \wedge b)$. Moreover, $(a \supset b) \wedge (b \supset a) = (a \vee b) \supset (a \wedge b)$.

(iv) $I(a,b)$ implies $a \wedge (a \supset b) = a \wedge b$.

(v) $I(a,b)$ implies $(a \vee b) \wedge (a \supset b) = b$.

Proof. (i) Note that if $I(a,c)$ and $I(b,c)$ hold, then by 2.1.5 we have aCc and bCc. It follows that $(a \vee b)Cc$. We will show that (1) and (2) of definition 2.1.1 hold for the pair $(a \vee b, c)$. Since $(a \vee b)Cc$, $(a \vee b) \wedge [(a \vee b)' \vee c] = (a \vee b) \wedge c \leq c$ so that (1) holds. To show that (2) holds, suppose that $(a \vee b) \wedge h \leq c$. Then $a \wedge h \leq c$ and $b \wedge h \leq c$. Since $I(a,c)$ and $I(b,c)$, we obtain that $h \leq a' \vee c$ and $h \leq b' \vee c$. Hence, $h \leq (a' \vee c) \wedge (b' \vee c) = (a' \wedge b') \vee c = (a \vee b)' \vee c$. Finally, $(a \supset c) \wedge (b \supset c) = (a' \vee c) \wedge (b' \vee c) = (a \vee b)' \vee c = (a \vee b) \supset c$.

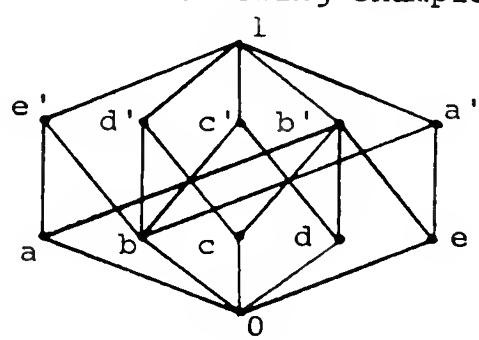
(ii) Suppose that $I(a,b)$ and $I(a,c)$. Then aCb and aCc so that $aC(b \wedge c)$. As in part (i), we check (1) and (2) of definition 2.1.1. $a \wedge [a' \vee (b \wedge c)] = a \wedge b \wedge c \leq b \wedge c$ so that (1) holds. For (2), suppose $a \wedge h \leq b \wedge c$. Then $a \wedge h \leq b$ and $a \wedge h \leq c$. Since $I(a,b)$ and $I(a,c)$, we conclude that $h \leq (a' \vee b) \wedge (a' \vee c) = a' \vee (b \wedge c)$. Finally, $(a \supset b) \wedge (a \supset c) = (a' \vee b) \wedge (a' \vee c) = a' \vee (b \wedge c) = a \supset (b \wedge c)$.

(iii) Note that $I(a,a)$ and $I(b,b)$ hold. Thus, by (i), $I(a \vee b,b)$ and $I(a \vee b,a)$. Now apply (ii) to these and obtain $I(a \vee b,a \wedge b)$. The rest is simple computation.

(iv) and (v) are direct computations.

Remark 2.1.7. (i) It is reasonable to ask whether or not the conclusions to parts (i) and (ii) of theorem 2.1.6 can be replaced by $I(a \wedge b,c)$ and $I(a,b \vee c)$ respectively. The answer is yes and the proof depends upon consequences of our next theorem.

(ii) A reasonable conjecture is that $I(e,f)$ is equivalent to $I(f',e')$. This conjecture is false, however, as the following example shows.



The above lattice is orthomodular, $I(b',e)$ holds, and $I(e',b)$ does not. In 2.1.18 we will give a necessary and sufficient condition for both $I(e,f)$ and $I(f',e')$ to hold.

Theorem 2.1.8. Let $e,f \in L$. Then $I(e,f)$ if and only if eCf and $e' \nabla (e \wedge f')$.

Proof. Suppose $I(e,f)$. Then by lemma 2.1.5, eCf. To show that $e' \nabla (e \wedge f')$, we use part (viii) of

theorem 1.4.5. Suppose $g \in L$ and g has the property $e \wedge f' \leq e' \vee g$. Then $e \wedge g' \leq e' \vee f$. Hence, $e \wedge g' \leq e \wedge (e' \vee f) = e \wedge f \leq f$. But $e \wedge g' \leq f$ and $I(e,f)$ imply $g' \leq e' \vee f$. Therefore, $e \wedge f' \leq g$.

Conversely, suppose eCf and $e' \nabla (e \wedge f')$. Since eCf we have $e \wedge (e' \vee f) = e \wedge f \leq e$; hence (1) of 2.1.1 holds. Now suppose that $e \wedge h \leq f$. Then $e' \vee h' \geq f'$. Thus $f' \wedge e \leq e' \vee h'$. By part (viii) of theorem 1.4.5, we obtain $f' \wedge e \leq h'$. Thus $h \leq e' \vee f$ and so (2) of 2.1.1 holds.

<u>Corollary</u> 2.1.9. $a \nabla b$ if and only if a and b are orthogonal and $I(a',b')$. Hence, a orthogonal to b implies that $I(a',b')$ if and only if $I(b',a')$.

<u>Proof</u>. If a and b are orthogonal, then $a' \geq b$. By 2.1.8, $I(a',b')$ implies that $a \nabla (a' \wedge b)$. Therefore $a \nabla b$.

Conversely, if $a \nabla b$, then a and b are orthogonal. Therefore $a \nabla (a' \wedge b)$. By 2.1.8, $I(a',b')$.

<u>Corollary</u> 2.1.10. $a,b \in L$, $a \leq b$ imply $I(a,b)$.

<u>Proof</u>. $a \leq b$ implies $a \wedge b' = 0$. Since $x \nabla 0$ for any $x \in L$, in particular $a' \nabla 0$. Thus $a' \nabla (a \wedge b')$. By 2.1.8, $I(a,b)$.

<u>Corollary</u> 2.1.11. In $\bar{L}(H)$[1], $I(a,b)$ if and only if $a = 1$ or $a \leq b$.

---

[1] $\bar{L}(H)$ = the lattice of closed subspaces of a Hilbert space H.

Proof. In $\bar{L}(H)$, $x \nabla y$ if and only if $x = 0$ or $y = 0$. By 2.1.8 then, $I(a,b)$ is equivalent to aCb and either $a' = 0$ or $a \wedge b' = 0$. Thus, $I(a,b)$ is equivalent to $a = 1$ or aCb and $a \wedge b' = 0$. But aCb and $a \wedge b' = 0$ is equivalent to $a \leq b$.

Lemma 2.1.12. Let $e,f \in L$. Then the following are equivalent.

(i) $I(e,f)$.

(ii) $I(e' \vee f,e)$ and eCf.

(iii) $I(e,e' \vee f)$ and eCf.

(iv) $I(e,e \wedge f)$ and eCf.

Proof. We use theorem 2.1.8 and the fact that ∇ is symmetric. $I(a,b)$ is equivalent to $a' \nabla (a \wedge b')$ and aCb. This is then equivalent to $(a \wedge b') \nabla a'$ and aCb. But applying 2.1.8 we see that this last statement is equivalent to $I(a' \vee b,a)$ and aCb. This gives (i) equivalent to (ii). To obtain (ii) equivalent to (iii), let $a = e' \vee f$ and let $b = e$. Finally, (ii) equivalent to (iv) follows by setting $a = e$ and $b = e \wedge f$.

Theorem 2.1.13. Let $a,b \in L$.

(i) $I(a,b)$ and $I(a,c)$ imply $I(a,b \vee c)$. Moreover, $(a \supset b) \vee (a \supset c) = a \supset (b \vee c)$.

(ii) $I(a,c)$ and $I(b,c)$ imply $I(a \wedge b,c)$. Moreover, $(a \supset c) \vee (b \supset c) = (a \wedge b) \supset c$.

Proof. (i) By 2.1.12, $I(a,b)$ and $I(a,c)$ imply that $I(a' \vee b,a)$ and $I(a' \vee c,a)$. Applying theorem 2.1.6,

part (i), we obtain $I((a' \vee b) \vee (a' \vee c),a)$, i.e., $I(a' \vee (b \vee c),a)$. Since $aC(b \vee c)$, we again apply 2.1.12 and obtain $I(a,b \vee c)$. The rest is straight forward computation.

(ii) $I(a,c)$ and $I(b,c)$ imply $I(a' \vee c,a)$ and $I(b' \vee c,b)$. Also, $I(a,c)$ and $I(b,c)$ imply aCc and bCc, hence $(a \wedge b)Cc$. We claim that $I(a' \vee b' \vee c, a \wedge b)$. Since $(a \wedge b)Cc$, it is clear that (1) of definition 2.1.1 holds. To check (2) suppose that $(a' \vee b' \vee c) \wedge x \leq a \wedge b$. It follows that $(a' \vee c) \wedge x \leq a$ and $(b' \vee c) \wedge x \leq b$. But since $I(a' \vee c,a)$ and $I(b' \vee c,b)$, we obtain $x \leq (a' \vee c)' \vee a = (a \wedge c') \vee a = (a \wedge c') \vee a = a$ and $x \leq (b' \vee c)' \vee b = (b \wedge c') \vee b = b$. Thus $x \leq a \wedge b$ and so (2) of definition 2.1.1 holds. Hence, $I(a' \vee b' \vee c,a \wedge b)$, i.e., $I((a \wedge b)' \vee c,a \wedge b)$. This and the fact that $(a \wedge b)Cc$ imply by 2.1.12 that $I(a \wedge b,c)$. The rest is straight forward computation.

<u>Definition</u> 2.1.14.

(i) $A(a) = \{x : x \in L \text{ and } I(a,x)\}$.

(ii) $P(a) = \{x : x \in L \text{ and } I(x,a)\}$.

<u>Corollary</u> 2.1.15. For every $a \in L$, $A(a)$ and $P(a)$ are sublattices of L.

<u>Theorem</u> 2.1.16. Let $a \in L$. Then the following statements are equivalent.

(i) $a \in C(L)$.

(ii) $I(a,x)$ for every $x \in L$.

(iii) $I(a,x)$ and $I(a,x')$ for some $x \in L$.

(iv) $I(a,a')$.

(v) $I(a,0)$.

(vi) $A(a) = L$.

(vii) $A(a)$ is a sub-orthomodular lattice of L.

Proof. (i) implies (ii). Let $x \in L$. Then for any $g \in L$ we have $(a' \vee g) \wedge ((a \wedge x') \vee g) = [a' \wedge ((a \wedge x') \vee g)] \vee g = (a' \wedge g) \vee g = g$. Hence, by part (vi) of 1.4.5 we have $a' \nabla (a \wedge x')$. Since $a \in C(L)$, aCx. Therefore by 2.1.8 we have $I(a,x)$.

(ii) implies (iii) is clear.

(iii) implies (i). $I(a,x)$ implies $a' \nabla (a \wedge x')$; $I(a,x')$ implies $a' \nabla (a \wedge x)$. By (iii) of 1.4.6 we then have $a' \nabla (a \wedge x') \vee (a \wedge x)$, i.e., $a' \nabla a$. Thus $I(a,x)$ and $I(a,x')$ imply that $a \in C(L)$.

(ii) implies (iv) is clear.

(iv) implies (v). This follows from lemma 2.1.12, parts (iv) and (i).

(v) implies (i). $I(a,0)$ implies $a' \nabla (a \wedge 0)$, i.e., $a' \nabla a$.

(i) implies (vi). This follows from the fact that (i) implies (ii) and (ii) implies (vi).

(vi) implies (vii) is clear.

(vii) implies (i). This follows from the fact that (vii) implies (iii) and (iii) implies (i).

Corollary 2.1.17. $C(L) = \cap\{P(x) : x \in L\}$.

Corollary 2.1.18. $I(e,f)$ and $I(f',e')$ both hold if and only if eCf and $e' \vee f \in C(L)$.

Proof. If $I(e,f)$ and $I(f',e')$, then we have by 2.1.8 that eCf, $e' \nabla (e \wedge f')$, and $f \nabla (f' \wedge e)$. By (iii) of 1.4.6 we obtain $(e' \vee f) \nabla (e \wedge f')$, i.e., $e' \vee f \in C(L)$.

Conversely, suppose that $(e' \vee f) \in C(L)$ and eCf. By parts (i) and (ii) of 2.1.16 we obtain $I(e' \vee f,e)$ and $I(e' \vee f,f')$. By parts (i) and (ii) of 2.1.12 we obtain $I(e,f)$ and $I(f',e')$.

2. Weakly Implicative Pairs.

By strengthening the hypothesis of (2) of definition 2.1.1 and assuming a priori that $g = e' \vee f$, we can define a weaker form of implicativity than $I(e,f)$.

Definition 2.2.1. We say that e and f form a weakly implicative pair, written $W(e,f)$, if and only if

(1) $e \wedge (e' \vee f) \leq f$

(2) $e \wedge f \leq e \wedge h \leq f$ imply $h \leq e' \vee f$,

both hold.

Remark 2.2.2. (i) $I(e,f)$ implies $W(e,f)$.

(ii) Definition 2.2.1 is equivalent to the following definition: $W(e,f)$ if and only if

(1) $e \wedge (e' \vee f) \leq f$,

(2) $e \wedge f = e \wedge h$ implies $h \leq e' \vee f$,

both hold.

(iii) Many of the theorems which are true

for implicative pairs do not hold for weakly implicative pairs, hence we will not study weakly implicative pairs in any detail. One theorem of consequence which does hold, however, is the analog of theorem 2.1.8. We state this without proof, noting that the proof makes use of part (viii) of theorem 1.4.7.

Theorem 2.2.3. Let $e,f \in L$. Then $W(e,f)$ if and only if eCf and $e'S(e \wedge f')$.

3. *Irreducibility Conditions*.

As we pointed out in chapter I, a lattice L is *irreducible* if and only if $C(L) = \{0,1\}$. In section 3 of that same chapter we saw that in the case of an atomic orthomodular lattice, one can give a condition stronger than irreducibility, namely, the *atomic bisection property*. In this section we define and investigate conditions which are stronger than irreducibility and which make sense in any orthomodular lattice.

Definition 2.3.1. (i) L is said to be *hyper-irreducible*, abbreviated H.I., if and only if the following hold.

(1) $L \neq 2^2$.[2]

(2) $0 < e < f < 1$ implies that there exists $g \in L$ such that eCg and $f \not C g$.

---

[2] 2^2 is used to denote the lattice  .

(ii) L is said to be weakly hyper-irreducible, abbreviated W.H.I., if and only if the following hold.

(1) $L \neq 2^2$.

(2) If $g \in L$, $g \neq 0$, $g \neq 1$, and g not an atom, then there exists $f \in L$ such that $f \wedge g \neq 0$ and $f \not{C} g$.

(iii) L is said to satisfy <u>condition (I)</u> if and only if there does not exist $e, f \in L$ such that $0 < e < f < 1$ and $I(f,e)$.

(iv) L is said to satisfy <u>condition (W)</u> if and only if there does not exist $e, f \in L$ such that $0 < e < f < 1$ and $W(f,e)$.

<u>Lemma</u> 2.3.2. H.I. implies W.H.I. and W.H.I. implies irreducibility.

<u>Proof</u>. <u>H.I. implies W.H.I.</u> Let $f \in L$ be such that $f \neq 0$, $f \neq 1$, and f is not an atom. Then there exists an h different from 0 and 1 such that eCh and $f \not{C} h$. Let $g = e \vee h$. $g \wedge f \wedge (e \vee h) \wedge f \leq e \wedge f = e$ so that $g \wedge f \neq 0$. It remains to show that $g \not{C} f$. Suppose gCf. Then $g\emptyset_f = g \wedge f = (e \vee h) \wedge f \wedge (e \wedge f) \vee (h \wedge f) = e \vee (h \wedge f)$. Also $g\emptyset_f = (e \vee h)\emptyset_f = e\emptyset_f \vee h\emptyset_f \wedge e \vee h\emptyset_f$. Therefore, $e \vee (h \wedge f) = e \vee h\emptyset_f$. It follows that $[e \vee (h \wedge f)] \wedge h\emptyset_f = h\emptyset_f$. Now eCh and eCf imply that $eC(h\emptyset_f)$. Thus $h\emptyset_f = (e \wedge h\emptyset_f) \vee (h \wedge f \wedge h\emptyset_f) = (e \wedge h\emptyset_f) \vee (h \wedge f) = [e \wedge f \wedge (h \vee f')] \vee (h \wedge f) = (e \wedge h) \vee (h \wedge f) = h \wedge f$. Hence $h\emptyset_f = h \wedge f$, i.e., hCf. This is

a contradiction, hence $g \not C f$.

W.H.I. implies irreducibility. Suppose L is reducible and W.H.I. We claim that there exists $f \in L$ such that f is different from 0 and 1, $f \in C(L)$, and f is not an atom. For, suppose not. Then $x \in C(L)$, $x \neq 0$ or 1 implies that x is an atom. Now, by reducibility there exists a g such that $0 < g < 1$ and $g \in C(L)$. Also, $g' \in C(L)$. But by assumption, g and g' are both atoms and so we have $L \simeq L(0,g) \times L(0,g') = 2^2$, a contradiction. Hence, there exists $f \in C(L)$ such that $0 < f < 1$ and f is not an atom. But W.H.I. implies that there exists an h such that $f \not C h$. But since $f \in C(L)$, this is a contradiction. Hence, W.H.I. implies irreducibility.

Theorem 2.3.3. Let L be atomic. Then L is H.I. if and only if L has the atomic bisection property.

Proof. Let a and b be orthogonal atoms, $a \neq b$. Then $0 < a < a \vee b$. If $a \vee b = 1$, then since $1 \neq 2^2$, there exists an atom c such that $c \neq a$ and $c \neq b$. (Clearly $c < 1 = a \vee b$.) Hence we may suppose that $0 < a < a \vee b < 1$. This is equivalent to $0 < a' \vee b' < a' < 1$. By H.I. there exists an element $g \in L$ such that $gC(a' \wedge b')$ and $g \not C a'$. Now $g \nleq a' \wedge b'$ or else gCa'. Therefore we have $g\emptyset_{a \vee b} \neq 0$. Let $c \leq g\emptyset_{a \vee b}$ be an atom. If $c = b$, then since $gC(a \vee b)$ we have $b = c \leq g \wedge (a \vee b)$. Thus $b \leq g$ and so bCg. But bCg, $a' \wedge b'Cg$ imply $a'Cg$, a contradiction. Thus $c \neq b$. Similarly, $c \neq a$.

Conversely, suppose that $0 < e < f < 1$. Let a and b be atoms such that $a \leq f'$ and $b \leq e' \wedge f$. (Note $e' \wedge f \neq 0$ or else $e = f$.) By the atomic bisection property, there exists an atom c such that $c \leq a \vee b$, $c \neq a$, and $c \neq b$. Suppose $c \leq e' \wedge f$. Then, noting that a and b are orthogonal and hence commute, we have $c \leq (e' \wedge f) \wedge (a \vee b) = (e' \wedge f \wedge a) \vee (e' \wedge f \wedge b) = b$. Thus $c = b$, a contradiction. Therefore, $c \nleq e' \wedge f$. Suppose now that $c \leq f'$. Since aCb we have $c \leq f' \wedge (a \vee b) = (f' \wedge a) \vee (f' \wedge b) = a$. Hence $c = a$, a contradiction. Therefore, $c \nleq f'$. We claim that eCc and $f\not{C}c$. It is clear that eCc since $c \leq e'$. Suppose that fCc. Then $c\emptyset_f = c \wedge f = 0$ or c. If $c\emptyset_f = 0$, then $c \leq f'$, a contradiction. If $c \wedge f = c$, then $c \leq f$. But $c \leq e'$ so this would imply $c \leq e' \wedge f$, a contradiction. Thus $c\not{C}f$.

<u>Lemma</u> 2.3.4. Suppose $L \neq 2^2$. Then W.H.I. if and only if it has the following property: If $h \in L$ with $h \neq 0$, $h \neq 1$, and h' not an atom, then there exists a complement k of h such that $k \neq h'$ and $k \wedge h' \neq 0$.

<u>Proof</u>. Suppose that L is W.H.I. Since $h' \neq 0$, $h' \neq 1$, and h' is not an atom, there exists $f \in L$ such that $h' \wedge f \neq 0$, $h\not{C}f$. Put $k = [(h' \vee f') \wedge f] \vee (h' \wedge f')$. Note that $k \wedge h' = (f \wedge h') \vee (h' \wedge f') \geq h' \wedge f$ so that $k \wedge h' \neq 0$. If $k = h'$, then $h' = (f \wedge h') \vee (h' \wedge f')$ which implies that hCf, a contradiction. Thus $k \neq h'$. Finally we note: $k \vee h = [(h' \vee f') \wedge f] \vee [(h' \wedge f') \vee h] = [(h' \vee f') \vee (h' \wedge f') \vee h] \wedge [f \vee (h' \wedge f') \vee h] = 1 \wedge$

$[(f \vee h) \vee (f \vee h)'] = 1$ and $k \wedge h = (h' \vee f') \wedge (f \wedge h) = 0$.

Conversely, suppose that for $h \in L$ with $h \neq 0$, $h \neq 1$, and h' not an atom, there exists $k \in L$ such that $g' \vee k = 1$, $g' \wedge k = 0$, $g \neq k$, and $k \wedge g \neq 0$. If gCk we would have $g' \wedge k' = (g' \vee k) \wedge k' = k'$, and so $g \leq k$. But then, since $g' \wedge k = 0$, we would have $g = k$. This is a contradiction, thus $g \not{C} k$.

Lemma 2.3.5. Let $L \neq 2^2$. Then condition (W) implies condition (I) and condition (I) implies irreducibility.

Proof. Condition (W) implies condition (I) by remark 2.2.2. To show condition (I) implies irreducibility we will show the contrapositive. Suppose L is reducible. Then this together with the fact that $L \neq 2^2$ imply that there exists $f \in C(L)$ such that $f \neq 0$, $f \neq 1$, and f is not an atom. Let e be chosen so that $0 < e < f < 1$. By theorem 2.1.16, $I(f,e)$.

Theorem 2.3.6. L is hyper-irreducible if and only if condition (W) holds.

Proof. We first claim that if $0 < e < f < 1$, then $C(e) \subset C(f)$ is equivalent to $W(f,e)$. For $W(f,e)$ if and only if eCf and $f'S(f \wedge e')$. But $f'S(f \wedge e')$ is equivalent to the condition $gC(f' \vee (f \wedge e'))$ implies gCf' and $gC(f \wedge e')$ by Theorem 1.4.9, part (vii). Since $0 < e < f < 1$, this last condition is equivalent to gCe'

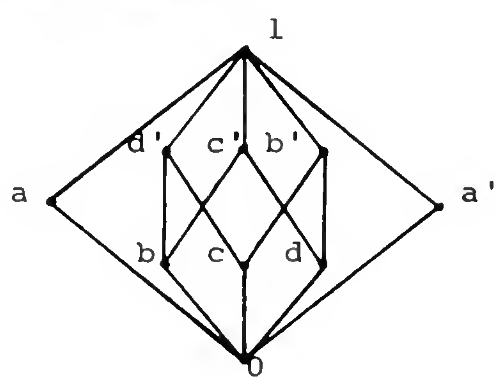
implies gCf', i.e. $C(e) \subset C(f)$. The theorem now follows by noting that L is hyper-irreducible if and only if $0 < e < f < 1$ implies $C(e) \not\subset C(f)$.

<u>Corollary</u> 2.3.7. If $\nabla = S$, then L is hyper-irreducible if and only if condition (I) holds. In general, hyper-irreducibility implies condition (I).

<u>Theorem</u> 2.3.8. If L is irreducible and modular, then L is W.H.I.

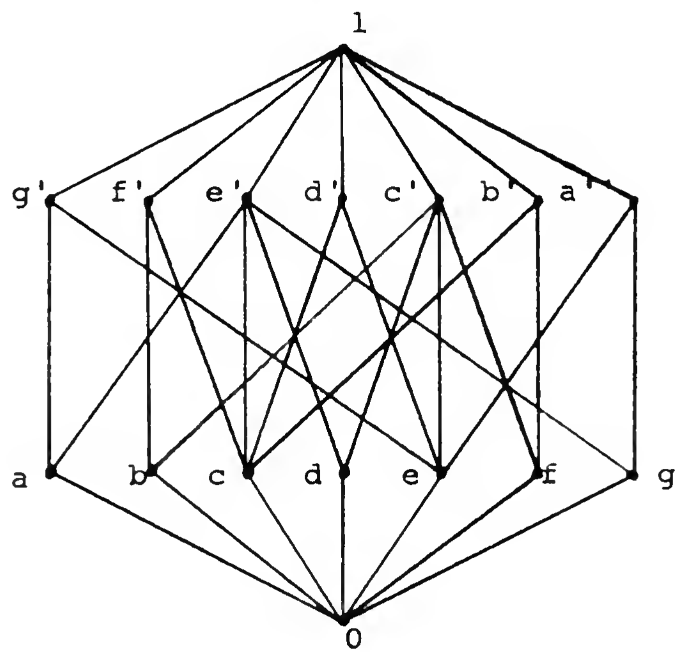
<u>Proof</u>. Suppose $g \in L$, $g \neq 0$, $g \neq 1$, and g is not an atom. Since L is irreducible, then there exists $f \in L$ such that $f \neq g$ and f is a complement of g'. If fCg then $f = g$. Hence we must have $f \not{C} g$. If $f \wedge g \neq 0$ we are done. Hence suppose $f \wedge g = 0$. Since g is not an atom, there exists $c \in L$ such that $0 < c < g$. Then $(c \vee f) \wedge g = c \vee (f \wedge g) = c$ so that $(c \vee f) \wedge g \neq 0$. If $(c \vee f)Cg$, then $c = (c \vee f) \wedge g = (c \vee f \vee g') \wedge g = g$. This is a contradiction, hence $(c \vee f) \not{C} g$. Thus $c \vee f$ will work.

<u>Example</u> 2.3.9. The following is an example of an orthomodular lattice in which property (I) holds (and hence is an irreducible lattice) but which is not W.H.I.



It is a simple matter to see that (2) of definition 2.1.1 fails for all pairs (f,e) where $0 < e < f < 1$. Thus condition (I) holds. To show that L is not W.H.I. we can use lemma 2.3.4. For, a and a' are the only complements of b other than b', but $a \wedge b' = a' \wedge b' = 0$.

Example 2.3.10. The following is an example of an orthomodular lattice that is W.H.I. but in which condition (I) fails. This lattice was first given by Dilworth and thus is usually denoted by the symbol D_{16}.



We can use lemma 2.3.4 to show that D_{16} is W.H.I. For, the complements given in the following table satisfy that lemma.

element	complement
a	c'
b	d'
c	a'
d	b'
e	b'
f	d'
g	c'

On the other hand, $I(e',d)$ holds so that condition (I) fails. Note that since condition (I) fails, D_{16} is not hyper-irreducible.

Example 2.3.11. $\bar{L}(H)$, H a Hilbert space, is hyper-irreducible by corollary 2.1.11 and the fact that $\nabla = S$.

Remark 2.3.12. (i) The consequences of the above theorems and examples can be depicted in diagram of implication as follows:

$$\begin{tikzcd}
 & & \text{W.H.I.} \arrow[dr, Rightarrow] & \\
\text{cond.(W)} \arrow[r, Leftrightarrow] & \text{H.I.} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{irreducible} \\
 & & \text{cond. (I)} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

(ii) We can now extend theorem 1.3.7 to the following theorem.

Theorem 2.1.13. Let L be atomic. Then the following are mutually equivalent.

(i) L has the atomic bisection property.

(ii) L is hyper-irreducible.

(iii) Every interval $L(e,f)$ is irreducible.

(iv) Every interval $L(0,a)$ is irreducible.

If in addition L is complete and modular, we can add:

(v) L is irreducible.

(vi) L is weakly hyper-irreducible.

(vii) Condition (I) holds for L.

(viii) Condition (W) holds for L.

CHAPTER III

ADDITIONAL RESULTS

In the last chapter it was clear that our study of implicativity was intimately associated with ∇ and S. Thus it is reasonable to have a brief look at some theorems which are involved with the general problem of finding exactly when ∇ and S are equal.

1. <u>The relative center property</u>.

<u>Definition</u> 3.1.1. (i) L is said to have the <u>relative center property</u> providing that for any $a \in L$, e is central in $L(0,a)$ if and only if $e = a \wedge z$ for some $z \in C(L)$.

(ii) The <u>central cover of e</u> (when it exists) is defined to be the infimum of all of the central elements which are greater than e. The central cover of e is denoted by $e\gamma$.

(iii) $e^{\nabla} = \vee \{ x : x \nabla e \}$, whenever this supremum exists.

<u>Lemma</u> 3.1.2. Let $e, a \in L$. Then, $e = z \wedge a$ for some $z \in C(L)$ if and only if $e \leq a \leq e \vee z'$, $e \leq z$, and $z \in C(L)$.

Proof. Let $z \in C(L)$ and suppose $e \leq a \leq e \vee z'$ and $e \leq z$. Then $e = e \wedge z \leq a \wedge z \leq (e \vee z') \wedge z = e \wedge z = e$.

Conversely, suppose that $e = z \wedge a$ for some $z \in C(L)$. Then $e' = a' \vee z'$. Thus $e' \wedge a = (a' \vee z') \wedge a = z' \wedge a$. $a = e \vee (a \wedge e') = e \vee (z' \wedge a) = (e \vee a) \wedge (z' \vee e) = a \wedge (z' \vee e)$. Therefore $e \leq a \leq e \vee z'$. $e \leq z$ is clear.

<u>Corollary</u> 3.1.3. If L has the relative center property and γ exists, then $e \in C(L(0,a))$ if and only if $e \leq a \leq e \vee (e\,\gamma)'$.

<u>Proof</u>. If $e \in C(L(0,a))$, then $e = z \wedge a$ for some $z \in C(L)$. For any such z, $e \leq z$. Therefore $e\,\gamma \leq z$. Hence, $e \leq a$ and $e \leq e\,\gamma$ imply that $e \leq a \wedge e\,\gamma \leq a \wedge z = e$. Therefore $e \in C(L(0,a))$ if and only if $e = (e\,\gamma) \wedge a$. The result now follows from the lemma.

<u>Theorem</u> 3.1.4. If L has the relative center property, then the following are equivalent.

(i) $a \nabla b$.

(ii) $a \wedge b = 0$ and there exists no $x \neq b$ such that $a \dot{\vee} x = a \dot{\vee} b$.

(iii) $a \wedge b = 0$ and a and b are central in $L(0, a \vee b)$.

(iv) aSb.

(v) There exists central elements z_a and z_b such that $a \leq z_a \wedge z_b'$ and $b \leq z_a' \wedge z_b$.

(vi) (If γ exists) $a\,\gamma \wedge b\,\gamma = 0$.

Proof. (i) implies (11). If $a \nabla b$ and $a \dot{\vee} x = a \dot{\vee} b$, then $x \wedge b = (a \vee x) \wedge b = (a \vee b) \wedge b = b$. Therefore $b \leq x$. Note that $x \in L(0, a \vee b)$. Now $a \nabla b$ implies a is orthogonal to b, whence $a^{\#} = b$ in $L(0, a \vee b)$. Now $b \leq x$ implies that aCx in $L(0, a \vee b)$. Therefore,
$b = b \wedge x = a^{\#} \wedge x = a^{\#}\emptyset_x = x \wedge (a^{\#} \vee x^{\#}) = x \wedge (a \wedge x)^{\#}$
$= x \wedge 0^{\#} = x$.

(ii) implies (iii) is clear.

(iii) implies (iv). Let $x \in L$ such that $x \leq a \vee b$. Then $x = (a \wedge x) \vee (b \wedge x)$. Hence aSb.

(iv) implies (v). aSb implies a and b are central in $L(0, a \vee b)$. Thus there exists central elements z_a and z_b such that $a = z_a \wedge (a \vee b)$ and $b = z_b \wedge (a \vee b)$. Then $z_b \wedge a = z_b \wedge z_a \wedge (a \vee b)$ and $z_a \wedge b = z_b \wedge z_a \wedge (a \vee b)$. But aSb implies that $a \wedge b = 0$, so $0 = a \wedge b = z_a \wedge z_b \wedge (a \vee b)$. Thus $z_b \wedge a = 0$ and $z_a \wedge b = 0$. Hence $z_b' \geq a$ and $z_b' \geq b$. It follows that $a \leq z_a \wedge z_b'$ and $b \leq z_a' \wedge z_b$.

(v) implies (i). If $b \leq z_a' \wedge z_b$ and $a \leq z_a' \wedge z_b$, then $a \wedge b = 0$. We will make use of theorem 1.4.5, part (vii). Let $x \wedge L$. Then $(z_a \vee x) \wedge b = (z_a \wedge b) \vee (x \wedge b) = x \wedge b$. Thus $x \wedge b \leq (a \vee x) \wedge b = (z_a \vee x) \wedge b \leq x \wedge b$. Therefore we have $(a \vee x) \wedge b = x \wedge b$. Thus $a \nabla b$.

(v) implies (vi). By (v), $a = a\gamma \wedge (a \vee b)$, $b = b\gamma \wedge (a \vee b)$. $0 = a \wedge b = a\gamma \wedge b\gamma \wedge (a \vee b)$ implies that $a\gamma \wedge b\gamma = 0$.

<u>(vi) implies (v)</u>. Merely take $z_a = a\gamma$ $z_b = b\gamma$.

<u>Lemma</u> 3.1.5. Let L have the relative center property. If either e^{∇} or $e\gamma$ exists for some e, then they both exist and $e^{\nabla} = (e\gamma)'$.

<u>Proof</u>. Suppose $e\gamma$ exists. $e \wedge (e\gamma)' = 0$ and for every $x \in L$, $(e \vee x) \wedge (e\gamma)' = x \wedge (e\gamma)'$. Thus by (vii) of 1.4.5 we have $e \nabla (e\gamma)'$. If $e \nabla f$, then by theorem 3.1.3, e and f are central in $L(0, e \vee f)$. Thus $e = (e\gamma) \wedge (e \vee f) = e \vee (e\gamma \wedge f)$. Hence, $e\gamma \wedge f \leq e$ implies $e\gamma \wedge f = e\gamma \wedge f \wedge e = 0$. Therefore, $f \leq (e\gamma)'$. Hence $e^{\nabla} = (e\gamma)'$.

Suppose e^{∇} exists. Then $e \nabla e^{\nabla}$, so e, e^{∇} are central in $L(0,e \vee e\ )$. Hence there exists $z \in C(L)$ such that $e^{\nabla} = z \wedge (e \vee e^{\nabla}) = (z \wedge e) \vee (z \wedge e^{\nabla}) = (z \wedge e) \vee e^{\nabla}$. Hence $z \wedge e \leq e^{\nabla}$ and so $z \wedge e = 0$. This result together with the fact that for all $x \in L$, $(e \vee x) \wedge z = x \wedge z$, implies that $e \nabla z$. Hence $z \leq e$. Thus we have shown that $z = e$. Since $e \wedge e^{\nabla} = 0$ and $e^{\nabla} \in C(L)$, we have that $e \leq (e^{\nabla})'$. If $z_1 \in C(L)$, $z_1 \geq e$, then $e \nabla z_1'$. (Use part (v) of theorem 1.4.5.) Thus $z_1' \leq e^{\nabla}$ and so $(e^{\nabla})' \leq z_1$. Hence $(e^{\nabla})' = e\gamma$.

<u>Lemma</u> 3.1.6. (Holland). Suppose e^{∇} and $e\gamma$ exist for $e \in L$ and moreover that $e^{\nabla} = (e\gamma)'$. Then if eSf implies $e \nabla f$, then L has the relative center property.

<u>Proof</u>. Let e be central in $L(0,a)$. Then

$a = e \dot{\nabla} e^{\#}$ and $e^{\#} = a \wedge e'$. Now e central in $L(0, e \vee e^{\#})$ implies eSe. By hypothesis then, $e \nabla e^{\#}$. Since $e^{\nabla} = (e\gamma)'$. Thus $e^{\#} \wedge (e\gamma) = 0$. Now $e \leq e\gamma \wedge a$ implies $e\gamma \wedge a = e \vee (e\gamma \wedge a \wedge e') = e \vee (e\gamma \wedge e^{\#}) = e$.

<u>Lemma</u>. 3.1.7. If b is central in $L(0,a)$, $z \in C(L)$, then $b \wedge z$ is central in $L(0, a \wedge z)$.

<u>Proof</u>. Let $x \in L(0, a \wedge z)$. Then $x \in L(0,a)$. $(b \wedge z)\emptyset_x = x \wedge ((b \wedge z) \vee x^{\#}) = x \wedge (b \vee x^{\#}) \wedge (z \vee x^{\#}) = x \wedge (b \vee (x' \wedge a \wedge z)) \wedge (z \vee x^{\#}) = x \wedge (b \vee (x' \wedge a)) \wedge (b \vee z) \wedge (z \vee x^{\#}) = (x \wedge b) \wedge (b \vee z) \wedge (z \vee x^{\#}) = (x \wedge b) \wedge (z \vee x^{\#}) = (x \wedge b \wedge z) \vee (x \wedge b \wedge x^{\#}) = x \wedge b \wedge z$. Therefore, $b \wedge z$ commutes with x for every $x \in L(0, a \wedge z)$.

<u>Theorem</u> 3.1.8. If γ exists for the lattice L, then the following statements are equivalent.

(i) L has the relative center property.

(ii) e is central in $L(0,a)$ if and only if $e \leq a \leq e \vee (e\gamma)'$.

(iii) e^{∇} exists for all e, $e^{\nabla} = (e\gamma)'$, and eSf implies $e \nabla f$.

(iv) $a \wedge b = 0$, $a\gamma \wedge b\gamma \neq 0$ imply that there exists $x \neq b$ such that $a \dot{\vee} x = a \dot{\vee} b$.

<u>Proof</u>. (i) equivalent to (ii) was corollary 3.1.3.

<u>(i) implies (iii)</u> follows from lemma 3.1.5 and theorem 3.1.4.

<u>(ii) implies (i)</u> was lemma 3.1.6.

<u>(i) implies (iv)</u> was theorem 3.1.4.

<u>(iv) implies (i)</u>. Let e be central in $L(0,a)$. Suppose $e < e\gamma \wedge a$. Then e is central in $L(0, e\gamma \wedge a)$. Also, $e\gamma \wedge a = e \dot{\vee} e^{\#}$ where $e^{\#} = e\gamma \wedge a \wedge e' \neq 0$. Then $e^{\#}\gamma \wedge e\gamma \neq 0$. Hence there exists $x \neq e^{\#}$ such that $e \dot{\vee} x = e \dot{\vee} e^{\#}$. Therefore e is not central in $L(0, e\gamma \wedge a)$, a contradiction. So $e = e\gamma \wedge a$.

<u>Definition</u> 3.1.9. L is said to be <u>relatively irreducible</u> if and only if every interval $L(0,a)$ is irreducible.

<u>Theorem</u> 3.1.10. If $C(L)$ is atomic and γ exists, then L has the relative center property if and only if $L(0,z)$ is relatively irreducible for every atom $z \in C(L)$.

<u>Proof</u>. Suppose L has the relative center property. Let z be an atom in $C(L)$ and let $a \leq z$. If $b \in C(L(0,a))$, then $b = a \wedge x$ for some $x \in C(L)$. Now $b = b \wedge z = a \wedge x \wedge z$. But z an atom in $C(L)$ and $x \in C(L)$ imply that $x \wedge z = 0$ or z. Therefore, $b = a \wedge x \wedge z = 0$ or $a \wedge z$. But $a \wedge z = a$ so that $b = 0$ or $b = a$. Hence $L(0,a)$ is irreducible for every $a \in L(0,z)$.

Conversely, suppose e is central in $L(0,a)$. Then by lemma 3.1.7, e is central in $L(0, e\gamma \wedge a)$. By the same lemma, $e \wedge z$ is central in $L(0, e\gamma \wedge a \wedge z)$ for every atom $z \in C(L)$. By hypothesis, $e \wedge z = 0$ or

$e \wedge z = e\gamma \wedge a \wedge z$. If $e \wedge z = 0$ we have $(e \wedge z)\gamma =$ $e\gamma \wedge z = 0$ and so $e\gamma \wedge a \wedge z = 0$. Thus in any case we have $e \wedge z = e\gamma \wedge a \wedge z$ for every atom z in $C(L)$. Hence, $e = e \wedge 1 = e \wedge \vee\{z : z$ an atom in $C(L)\} = \vee\{e \wedge z :$ z an atom in $C(L)\} = \vee\{e\gamma \wedge a \wedge z :$ z an atom in $C(L)\} =$ $e\gamma \wedge a$.

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BIOGRAPHICAL SKETCH

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